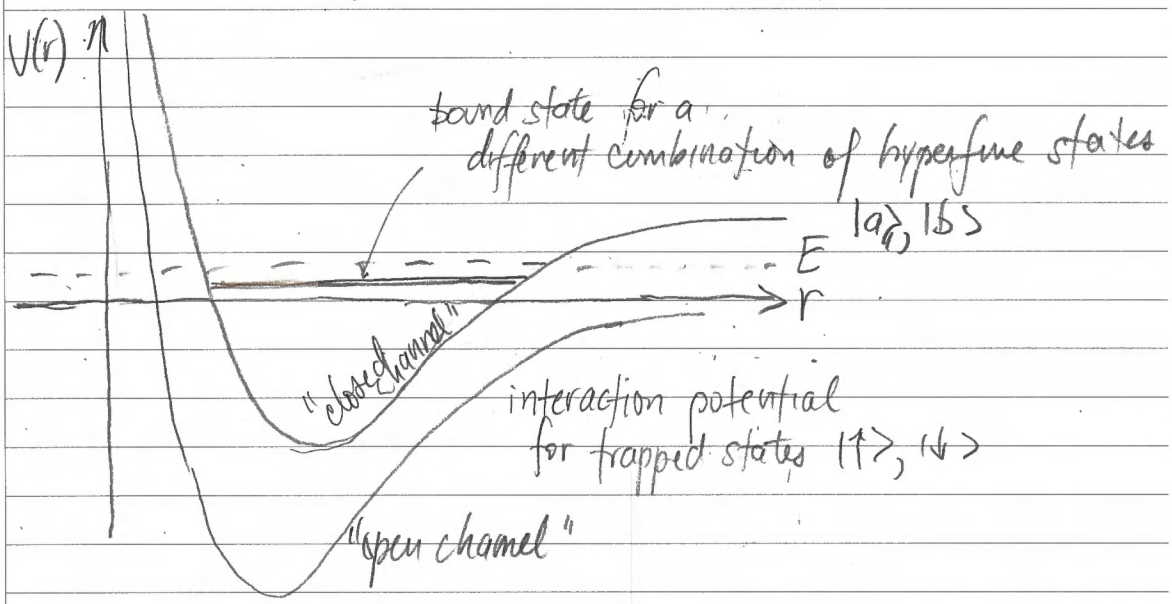
Experiments with strongly interacting Fermi gases

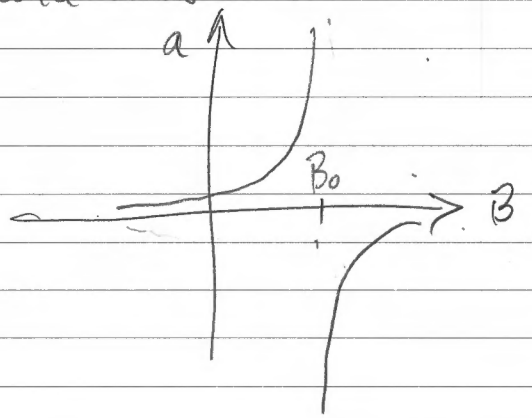
One way to make the interactions between atoms important: a Feshbach resonance

Zeeman energy difference depends on magnetic field B



$|\uparrow\rangle, |\downarrow\rangle$ and $|a\rangle, |b\rangle$ couple at short range because the interaction is not diagonal in the hyperfine basis

When a bound state in the closed channel matches the kinetic energy in the open channel, a Feshbach resonance occurs



$$a = a_{bg}\left(1 - \frac{\Delta B}{B - B_0}\right)$$

Feshbach resonances are a key tool for exploring the analogy of BCS pairing in a trapped gas.

In an electronic solid, BCS SC occurs because phonons give rise to an effective s-wave attraction btw electrons

Trapped gas:

$$T_c \approx 0.28\, T_F\, e^{-\frac{\pi}{2 k_F |a|}} \quad \text{(assuming } a < 0\text{)}$$

so, if a is small, then T_c is small

ex: $T_F = 1\,\mu K$ $\qquad \frac{\hbar^2 k_F^2}{2m} = E_F$

$a_{bg} = 150\, a_0$ $\qquad k_F = \frac{\sqrt{2mE_F}}{\hbar}$

$T_c = 0.02\,pK$

But, w/ $a \to 3000\, a_0$, $T_c \to 100\,nK$

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TOPIC:

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FILE UNDER:

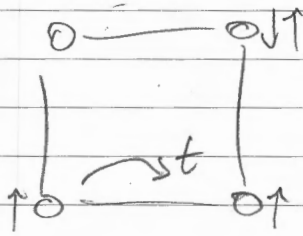
Hubbard models with optical lattices

Another route to strong interactions — use a lattice to suppress kinetic energy and enhance interaction energy

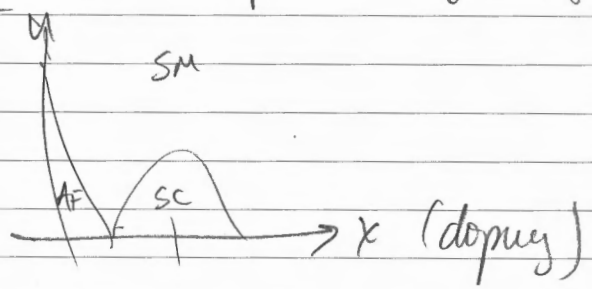
What is the Hubbard model? Simplest model of interacting electronic solids, inspired by high T_c cuprates

$$H = -t \sum_{\langle i,j \rangle, \sigma} \left(c^\dagger_{i,\sigma} c_{j,\sigma} + h.c. \right) + U \sum_i n_{i\uparrow} n_{i\downarrow} - \mu \sum_{i,\sigma} n_{i,\sigma}$$

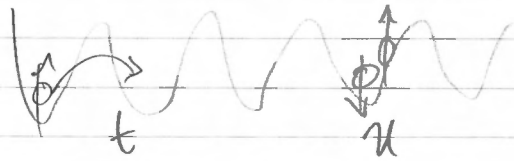
electrons localized to Cu atoms in planes, "Coulomb" interaction well screened



For $t \sim U$, the model is unsolved. Perturbation theory fails; QMC only possible for few 10s of particles. We do not know if this model can account for high-T_c phase diagram, for example.



We can exactly realize this model using fermionic atoms trapped in an optical lattice, as well as its bosonic counterpart using bosonic atoms.



atoms $\leftrightarrow$ electrons

optical lattice $\leftrightarrow$ crystalline lattice

*note - we phonons in optical lattice

To see how this works, start w/ the non-interacting, single particle problem.

Work in 1D -> the Hamiltonian is separable for cubic & square lattices, so 1D is easy to generalize.

Band structure

Optical lattice: $H = \frac{p^2}{2m} + V(x)$

$\lambda \Longrightarrow \Longleftarrow$

$V(x) = -V_0 \cos^2(k_L x)$ red-detuned lattice

$k_L = 2\pi/\lambda$

Bloch's theorem:

$\phi_q^{(l)}(x) = e^{iqx/\hbar} u_q^{(l)}(x)$ Bloch wavefunction

l: band index

If we plug $\phi_q^{(l)}(x)$ into the TISE, we get:

$$\left(\frac{(\hat{p}+q)^2}{2m} + V \right) u_q^{(l)} = \varepsilon_q^{(l)} u_q^{(l)}$$

To solve for the $u_q^{(\ell)}$ functions, we use a Fourier expansion

- first, notice that $V = -V_0 \cos^2(k_L x) = -V_0/2\,(1+\cos(2k_L x))$

$$= -\frac{V_0}{4}\left(e^{2ik_L x} + e^{-2ik_L x} + 2\right)$$

this means that u_q can only have Fourier components that are multiples of $0, \pm 2\hbar k_L, \pm 4\hbar k_L, \ldots$

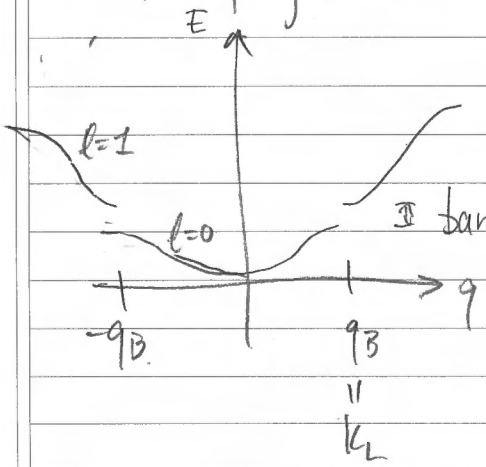
$$u_q^{(\ell)}(x) = \sum_{m=-\infty}^{\infty} c_m^{(\ell)}(q)\, e^{2im k_L x}$$

And then we solve for the $c_m^{(\ell)}$ using the eigenvalue equation:

$$\sum_{m=-\infty}^{\infty} H_{m,m'}\, c_m^{(\ell)}(q) = E_q^{(\ell)}\, c_{m'}^{(\ell)}(q)$$

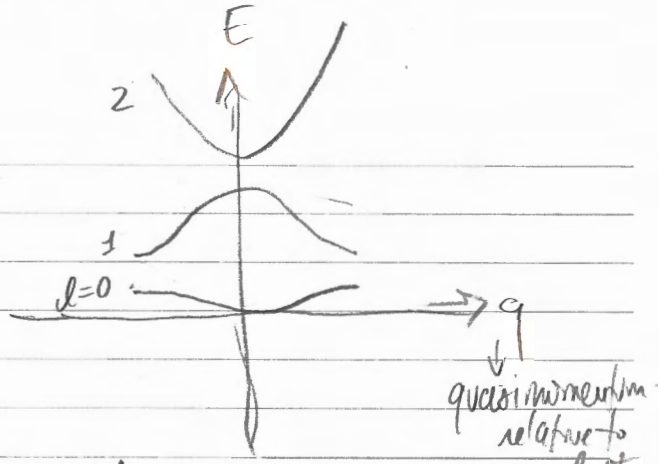
$$H_{m,m'} = \begin{cases} \left(2m + \frac{q}{2\hbar k_L}\right)^2 E_R, & m = m' \\ -V_0/4, & |m-m'| = 1 \\ 0, & \text{otherwise} \end{cases}$$

This must be solved numerically, typically by truncating the sum



∃ bandgap – w/ $q = k_L$, there are 2 standing wave solutions

typically shown in reduced-zone scheme:

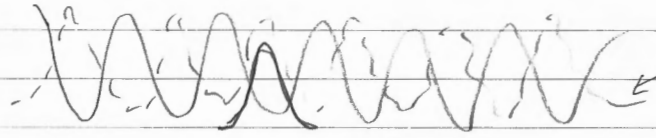


Experimental T and E_F small enough so that only lowest-energy band is occupied

Note how V_0 controls the band structure — kinetic energy scale shrinks for higher $V_0 \Rightarrow$, so we can always make t small & enhance role of interactions

So far, we have not seen the Hubbard Hamiltonian — this is just standard band structure. To reduce to the Hubbard Hamiltonian, use the Wannier basis, which are wavefunctions localized at the lattice sites:

Work within lowest band from now on



Wannier function localized at site i

$$W_i(x) \propto \int_{-\infty}^{\infty} e^{iq(x-x_i)} \phi_q(x)\, dx$$

$$\phi_q(x) \propto \sum_i e^{iqx/\hbar} W_i(x-x_i)$$

} like Fourier pairs

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Expand field operators

$$\hat{\psi} = \sum \hat{a}_i w_i(x)$$

$$H = \int \hat{\psi}^\dagger \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \right) \hat{\psi}\, dx$$

$$= \sum_{ij} a_i^\dagger a_j \int dx\; w_i^*(x) \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x) \right) w_j(x)\, dx$$

$$= \sum_i a_i^\dagger a_i \int dx\; w_i^*(x) H w_i(x) +$$

nearest neighbors $\longrightarrow \sum_{\langle ij \rangle} a_j^\dagger a_i \int dx\; w_j^*(x) H w_j(x) +$ other terms

tight binding approximation – ignore all "other terms" and account for only tunneling btw. nearest neighbors

also, $\varepsilon_i = \int dx\; w_i^*(x) H w_i(x)$ is the kinetic & potential energy to occupy a site – included in μ

then $H = -t \sum_{\langle ij \rangle} (a_j^\dagger a_i + h.c)$

$t = -\int dx\; w_j^*(x) \left(-\frac{\hbar^2}{2m} + V(x) \right) w_i(x)$ is the tunneling energy

In this limit, you can get a closed form for the dispersion $\epsilon(q)$:

write $|\psi\rangle = \sum_\ell e^{iq\ell a} |\ell\rangle$

$|\ell\rangle$ = particle in $W_\ell(x)$ a: lattice spacing

$$H|\psi\rangle = -t \sum_i |i+1\rangle\langle i| + |i-1\rangle\langle i| \sum_\ell e^{iq\ell a} |\ell\rangle$$

$$= -t \sum_{i\ell} |i+1\rangle \delta_{i\ell} + |i-1\rangle \delta_{i\ell} e^{iq\ell a}$$

$$= -t \sum_\ell \left[|\ell+1\rangle e^{iq\ell a} + |\ell-1\rangle e^{iq\ell a} \right]$$

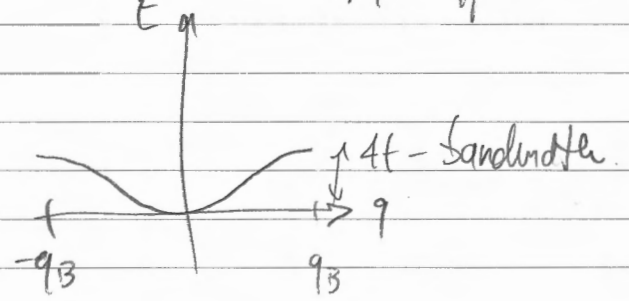
$$= -t \sum_j \left[|j\rangle e^{iq(j-1)a} + |j\rangle e^{iq(j+1)a} \right]$$

$$= -t \sum |j\rangle e^{iqja} 2\cos(qa)$$

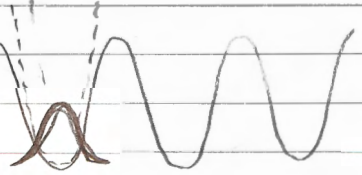
$$= -2t\cos(qa)\,|\psi\rangle$$

So, $\epsilon(q) = 2t(1-\cos(qa)) = 2t\left(1-\cos\left(\pi \frac{q}{q_B}\right)\right)$

add an energy offset so that $\epsilon(q=0)=0$



Also, if you approximate $w(x)$ as a gaussian — ground state of lattice well



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You can show

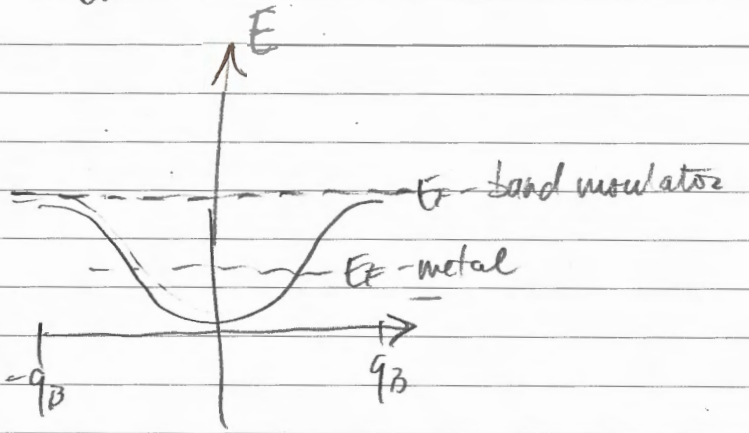
$$t \approx 8\pi^{3/2}\hbar^2 \frac{S^{3/4} e^{-2\sqrt{S}}}{m\lambda^2} \qquad S = V_0/E_R$$

* there are not good enough approximation for experiments

$$E_{bg} = 2E_R\sqrt{S}$$

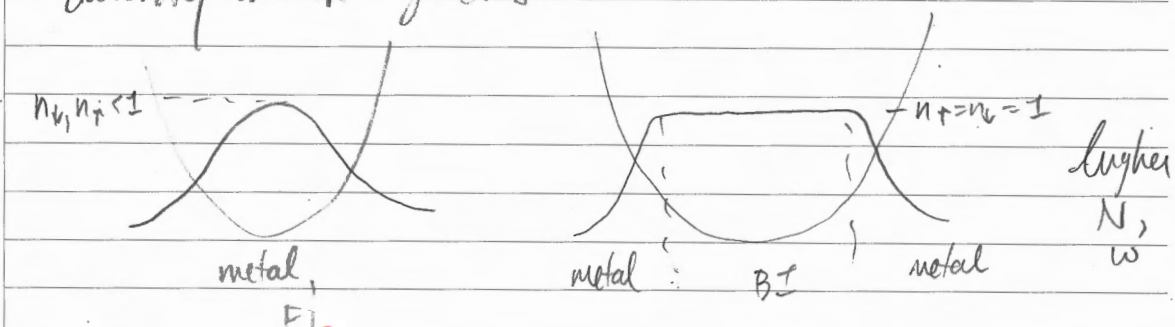
So, the lattice exponentially suppresses the tunneling

For this non-interacting problem, you get metallic and band-insulator states depending on E_F:



we can tune E_F using the trap and/or N

The trap complicates this story a bit, because the density is inhomogeneous!



band-gap prevents $n>1$ in the center

What about interactions?

Handle this with a contact potential, must work in 3D

$$H = \ldots + \frac{4\pi a\hbar^2}{m}\int d^3x_1 \int d^3x_2\, \psi^\dagger(\vec{x}_1)_\uparrow \psi^\dagger_\downarrow(\vec{x}_2)\, \delta(\vec{x}_1-\vec{x}_2)\, \psi_\uparrow(\vec{x}_1)\psi_\downarrow(\vec{x}_2)$$

$$= \ldots + \frac{4\pi a\hbar^2}{m}\sum_{ijlm}\int d^3x\, c^\dagger_{i\uparrow} c^\dagger_{j\downarrow} c_{l\uparrow} c_{m\downarrow}\, w^*(x-x_i)\, w^*(x-x_j)\cdot w(x-x_l)\, w(x-x_m)$$

built in Pauli-exclusion principle

make the approximation that only atoms on the same site interact: $i=j=l=m$

$$= \ldots + \frac{4\pi a\hbar^2}{m}\sum_i n_{i\uparrow} n_{i\downarrow}\int |w(x)|^4 d^3x$$

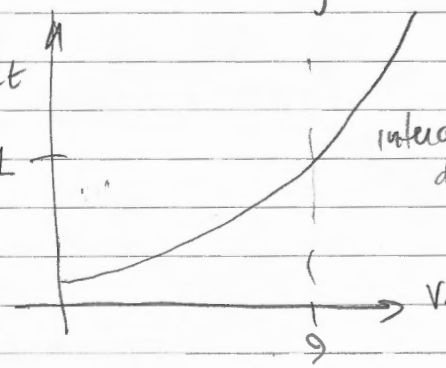
$$\equiv \ldots + \frac{4\pi a\hbar^2}{m} U \sum_i n_{i\uparrow} n_{i\downarrow}$$

In the deep / harmonic well approximation

$$U \simeq 8\sqrt{2}\,\pi^{5/2}\hbar^2 \frac{a s^{3/4}}{m\lambda^3}$$

compared w/ bands $U \sim 100\,nK\cdot k_B \sim 10\,peV$

So, by tuning the lattice intensity and V_0, we can tune the relative values of t and U



* in 3D, $\epsilon(q) = 2t(3 - \cos\pi q_x/q_B - \cos\pi q_y/q_B - \cos\pi q_z/q_B)$

the total kinetic energy that can be accumulated in the ground band is $12t$

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A very simplified, wrong version,

In the metallic state, particles are in Bloch wavefunctions - equally likely to be found on any site except for Pauli exclusion principle

So, if you look @ 1 site, you will find it doubly occupied 1/4 of the time

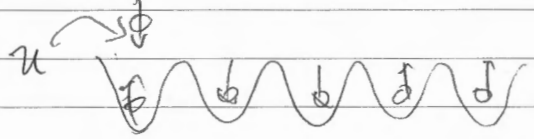
Interaction energy / particle is $U/4$ -

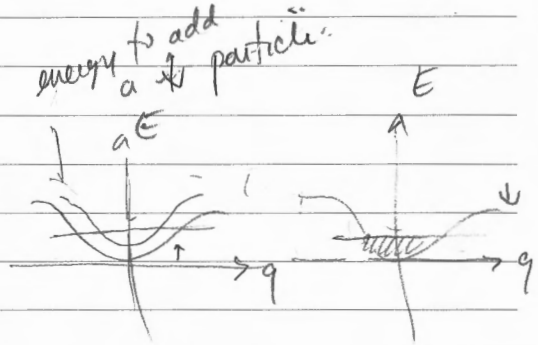
the average KE/particle for half-filled band is $3t$ (when $E_F = 6t$)

so, when $U/4 \geq 3t$, or $U > 12t$, it is energetically favorable for the particles to localize

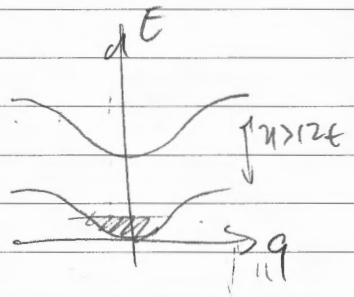
What is the Mott insulator?

Consider half-filling

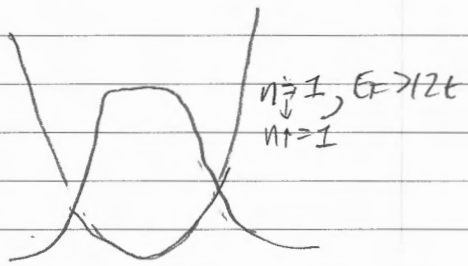




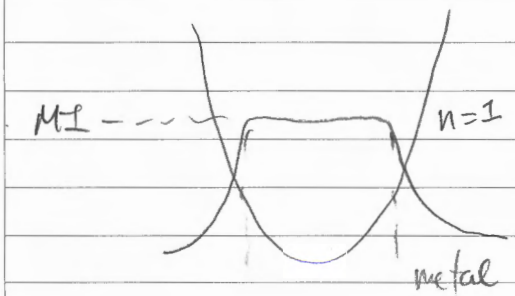
When $U > 12t$, a gap opens, and the kinetic energy of the particle cannot be accomodated - the system becomes insulating



Again, more complicated and a little different in the trap



$U > 12t$



@ higher density, a metal will form in the middle

↑ detected as signature of doubly-occupied sites

E-solinger - raise to 30 Er-

FR,

$5^2_{1/2}$ $5^2_{1/2}$

n